Ground state cooling of a nanomechanical resonator via parametric linear coupling

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(Received 23 March 2009; published 28 May 2009)

We present a cooling mechanism for a nanomechanical resonator that linearly couples with a LC oscillator. The linear coupling, when periodically modulated at red detuning, up-converts the low-frequency nanomechanical quanta to the high-frequency LC oscillator quanta which then dissipates in the circuits via the damping of the LC oscillator. We show that realistic circuit can be designed in the superconducting system to implement this scheme in the resolved-sideband regime and reach the quantum ground state of the nanomechanical mode. A detailed study of the origin of the quantum back-action noise in this cooling process is also presented.

DOI: 10.1103/PhysRevB.79.193407 PACS number(s): 85.85.+j, 07.10.Cm, 37.10.Vz, 85.25.−j

I. INTRODUCTION

Nanomechanical systems with ultrahigh quality factor (Q factor) have been demonstrated to approach the quantum limit in recent experiments.[1–5] The study of the quantum behavior in such systems can have profound impact on various topics including the detection of weak forces,[6] the study of classical-and-quantum boundary in macroscopic objects,[7,8] and quantum entanglement and quantum information.[9] Couplings between nanomechanical systems with solid-state devices[10] or atomic systems[11] have been widely studied and can facilitate the implementation of quantum control and quantum engineering in such systems.[12]

Ground state cooling is crucial for the quantum engineering of nanomechanical systems.[13] Recently, cooling of nanomechanical modes from room temperature to a few Kelvin has been achieved via dynamic back-action force or active feedback cooling in optical cavities and solid-state circuits[14–17] and the resolved-sideband regime has been demonstrated.[3,5] In theory, full quantum-mechanical treatments have been developed to study the cooling of nanomechanical modes via radiation pressure force and the dynamics during the cooling process.[18–24] It has also been proposed that cooling can be achieved by coupling the nanomechanical modes to solid-state two-level systems (qubits).[25,26] However, the quantum ground state has not been observed due to various experimental restrictions.

In this work, we study a cooling scheme that can be implemented in solid-state circuits and can reach the quantum limit of the nanomechanical mode with realistic experimental parameters. With the flexibility in device layout and parameter engineering, nanomechanical systems imbedded in solid-state circuits can have a broad range of parameters and diverse forms of coupling to other degrees of freedom such as oscillator modes and qubits. Here, using both a semiclassical approach and a full quantum-mechanical approach, we show that the Bogoliubov *linear* coupling between a nanomechanical resonator and a LC oscillator can be explored to cool the nanomechanical mode. The key to the underlying cooling mechanism is the "up-conversion" of the nanomechanical quanta to the LC oscillator quanta via the linear coupling.[27] An analog between this scheme and laser cooling for atomic systems can be drawn,[28] which reveals the origin of the quantum back-action noise.[29] The linear coupling can be implemented in a superconducting circuit by coupling the nanomechanical resonator capacitively with a superconducting resonator[30,31] and has been studied previously for generating entanglement and squeezed states in nanomechanical systems.[32] With the progress in several recent experiments,[5,13,14] it can be shown that the ground state can be reached in this scheme with current technology. Furthermore, the scheme does not require driving of the LC oscillator to high occupancy as does the scheme based on radiation pressure force.

II. CIRCUIT AND COUPLING

Consider the circuit in Fig. 1(a), where a nanomechanical resonator capacitively couples with the charge island of a LC oscillator. We denote the vibrational displacement of the nanomechanical mode as x and the annihilation operator as a, with $x=\delta x_0(a+a^\dagger)$ and x_0 being the quantum displacement. The coupling capacitance can be expressed as $C_x=C_{x0}+C_x' x$ to the first order of x, where $C_x'=\partial C_x/\partial x$ and the gate voltage is modulated by an external source v_c. The LC oscillator includes a capacitor C and an inductor L. To study the LC oscillator, we introduce a phase variable Φ for the charge island and its conjugate momentum P_Φ which describes the total charge on the island. Applying a Lagrangian approach for quantum circuits,[33] we derive the total Hamiltonian of the

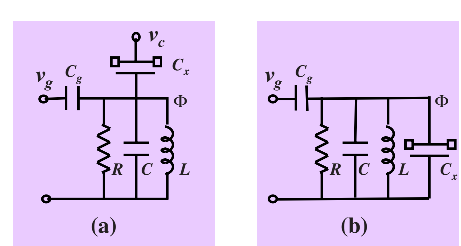

FIG. 1. (Color online) Schematic circuits of a nanomechanical resonator capacitively coupling with a LC oscillator. (a) Linear coupling and (b) radiation-pressure-like coupling. The resonator is indicated as a bar with two contacts.

coupled systems as $H_t = \hbar\omega_a a^\dagger a + H_c$, where ω_a is the frequency of nanomechanical mode and

$$H_c = \frac{(P_\Phi + v_c C'_x x)^2}{2(C_{\Sigma 0} + C'_x x)} + \frac{\Phi^2}{2L} \quad (1)$$

is the Hamiltonian of the LC oscillator including the coupling. Here, $C_{\Sigma 0} = C + C_g + C_{x0}$ is the total static capacitance. A gate voltage $v_g = -v_c C_{x0}/C_g$ is applied to a second gate capacitance C_g to balance the driving on the momentum P_Φ so that the average excitation in the LC oscillator is close to zero. We denote the frequency of the LC oscillator as $\omega_b = (LC_{\Sigma 0})^{-1/2}$ and the annihilation operator as b. Equation (1) can be written as

$$H_c = \hbar\omega_b b^\dagger b - g_r(a + a^\dagger)b^\dagger b - ig_l(a + a^\dagger)(b - b^\dagger), \quad (2)$$

which includes two coupling terms: a radiation-pressure-like coupling with the coupling constant

$$g_r = (\hbar\omega_b/2)(C'_x \delta x_0/C_{\Sigma 0}) \quad (3)$$

and a Bogoliubov linear coupling with the coupling constant

$$g_l = v_c\sqrt{\hbar\omega_b/2C_{\Sigma 0}}(C'_x \delta x_0) \quad (4)$$

that depends on the applied voltage v_c. For an rf voltage, the voltage source generates reactive power to pump the coupling periodically.

The ratio between the linear coupling and radiation-pressure-like coupling can be derived as $g_l/g_r = v_c\sqrt{2C_{\Sigma 0}/\hbar\omega_b}$, which is proportional to the applied voltage v_c. Given the typical parameters in a superconducting circuit (see below), we can have $g_l/g_r \sim 7\times 10^3$ at a voltage of $v_c = 300$ mV which requires a reactive power of ~ 850 μW. With nearly no excitation in the LC oscillator, the linear coupling is hence the dominant coupling in this system and we can neglect the other coupling term. For a constant driving voltage, Eq. (2) describes two coupled harmonic oscillators in thermal equilibrium at a given temperature. However, when the coupling is periodically modulated by an rf voltage source $v_c(t) = 2v_c \sin\omega_d t$ at a frequency ω_d, the scenario changes. The total Hamiltonian in the rotating frame of the driving frequency becomes

$$H_t^{rot} = \hbar\omega_a a^\dagger a - \hbar\Delta b^\dagger b + g_l(a + a^\dagger)(b + b^\dagger), \quad (5)$$

with a detuning $-\Delta = \omega_b - \omega_d$ and a constant coupling g_l. Here, we assume the driving is in the red-detuned regime with $-\Delta > 0$.

Damping is a crucial factor in the cooling process. We denote the damping rate of the nanomechanical mode as γ_m and the damping rate of the LC oscillator as κ_0, both of which are associated with a thermal bath at a temperature T_0 with the thermal occupation number $n_{i0} = [\exp(\hbar\omega_i/k_B T_0) - 1]^{-1}$ for $i = a, b$. At $T_0 = 20$ mK and a frequency of $\omega_b = 2\pi\times 7.5$ GHz for a superconducting LC oscillator, $n_{b0} \sim 10^{-8}$ and can be neglected.

III. SEMICLASSICAL THEORY

First, we apply a semiclassical circuit approach to explain the cooling process. Denote the voltage on the phase island (labeled as Φ in the circuit) as $v_b = \dot{\Phi}$. With a driving voltage $v_c e^{i\omega_d t}$ and a nanomechanical vibration $x(t) = xe^{i\omega_a t}$, we derive

$$v_b = \frac{-(\omega_d + \omega_a)^2 C'_x x v_c}{C_{\Sigma 0}[\omega_b^2 - (\omega_d + \omega_a)^2 + i\kappa_0(\omega_d + \omega_a)]}, \quad (6)$$

where the damping is $\kappa_0 = (RC_{\Sigma 0})^{-1}$, related with the resistance in the circuit.

The electromagnetic field inside the capacitor C_x generates a force on the nanomechanical resonator $F_e = \partial[C_x(v_c - v_b)^2/2]/\partial x$, where both C_x and v_b [as is shown in Eq. (6)] depend on the mechanical vibration x. It can be shown that $F_e \approx \lambda x - m\Gamma_c \dot{x}$ which contains a frictional force proportional to $\dot{x}$ besides a small modification to the elastic constant of the nanomechanical mode. The frictional force has a $\pi/2$-phase difference from the nanomechanical motion with $\dot{x} = i\omega_a x$ and can result in cooling of the nanomechanical motion. We find that the cooling rate is

$$\Gamma_c = \frac{4(g_l/\hbar)^2(\omega_d + \omega_a)^3\kappa_0/\omega_b}{[(\omega_d + \omega_a)^2 - \omega_b^2]^2 + (\omega_d + \omega_a)^2\kappa_0^2}, \quad (7)$$

as is plotted in Fig. 2. The cooling rate reaches maximum of $\Gamma_c \approx 4g_l^2/(\hbar^2\kappa_0)$ at the detuning $-\Delta \approx (\omega_a - \kappa_0^2/8\omega_b)$. Typically, with $\kappa_0 \ll \omega_a \ll \omega_b$, this means $-\Delta \approx \omega_a$.

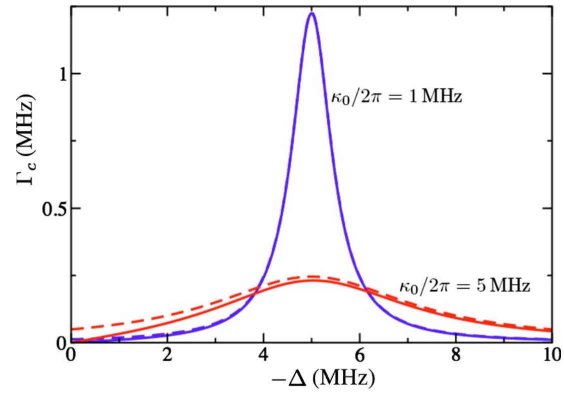

FIG. 2. (Color online) Cooling rate vs detuning. Solid curves are by the quantum theory in Eq. (10) and dashed curves are by the semiclassical theory in Eq. (7).

IV. QUANTUM THEORY

The semiclassical approach cannot explain the quantum back-action noise and the cooling limit for the system. Here, we apply the input-output theory to study the cooling process of this linearly coupled system. In the Heisenberg picture, we derive the following operator equations:

$$\dot{a} = -i\omega_a a - i(g_l/\hbar)(b + b^\dagger) - \frac{\gamma_m}{2}a + \sqrt{\gamma_m}a_{in}, \quad (8)$$

$$\dot{b} = i\Delta b - i(g_l/\hbar)(a + a^\dagger) - \frac{\kappa_0}{2}b + \sqrt{\kappa_0}b_{in}, \quad (9)$$

and their conjugate equations. Here, a_{in} and b_{in} are noise operators for the thermal bath of the corresponding modes.[27]

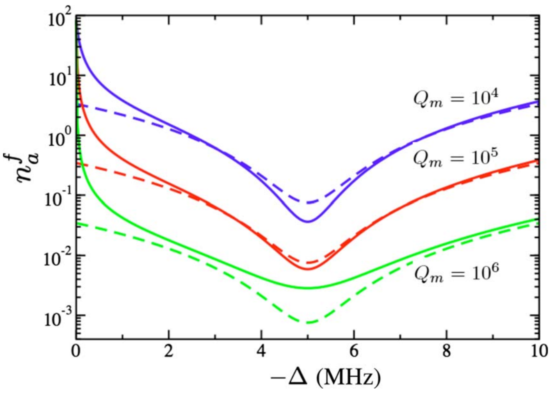

FIG. 3. (Color online) Occupation number n_a^f vs detuning. Solid curves are by the full quantum theory and dashed curves are when the counterrotating term is omitted.

Note that for radiation-pressure-like forces, linearization of the coupling relative to the amplitude of the driven cavity mode can result in a set of equations similar to Eqs. (8) and (9).[18,19,21,22,29] From these equations, the cooling rate can be derived as

$$\Gamma_c = \frac{4(g_l/\hbar)^2\kappa_0|\Delta|\omega_a}{\left(\Delta^2 - \omega_a^2 + \frac{\kappa_0^2}{4}\right)^2 + \omega_a^2\kappa_0^2}. \tag{10}$$

At $-\Delta \approx \omega_a(1-\kappa_0^4/128\omega_a^4)$ near the first red sideband, the cooling rate reaches the maximum with $\Gamma_c \approx 4g_l^2/[\hbar^2\kappa_0(1+\kappa_0^2/16\omega_a^2)]$. As can be seen in Fig. 2, in the resolve-sideband regime with $\kappa_0 \ll \omega_a$, the cooling rates from the semiclassical theory and the quantum theory are very close.

The stationary occupation number can be written as $n_a^f = (\Gamma_c n_0 + \gamma_m n_{a0})/(\Gamma_c + \gamma_m)$, where $n_0 \approx \kappa_0^2/16\omega_a^2$ results from the so-called quantum back-action noise.[23] When $\Gamma_c \gg \gamma_m$, $n_a^f \approx n_0 + (\gamma_m/\Gamma_c)(n_{a0} - n_0) \ll n_{a0}$ and the occupation number will be limited by n_0. Cooling of the nanomechanical mode can then be achieved in the resolve-sideband regime. In Fig. 3, we plot the occupation number versus detuning. It can be shown that the impact of the thermal occupation on the minimal occupation number decreases as the mechanical quality factor Q_m increases and the occupation number approaches n_0 at high Q_m.

The periodical modulation of the linear coupling is the key to this cooling scheme. For constant gate voltage, the cooling rate can be derived by replacing the detuning $-\Delta$ in Eq. (10) with the frequency ω_b,

$$\Gamma_c \approx 4g_l^2\kappa_0\omega_a/(\hbar^2\omega_b^3) \ll \gamma_m, \tag{11}$$

when $\omega_b \gg \omega_a$. It can also be shown that $n_a^f \approx n_{a0}$. Hence, at constant coupling, the LC oscillator together with its bath acts as an effective thermal bath at the temperature T_0.[34] For periodical modulation on the gate voltage, the effective frequency of the LC oscillator becomes $-\Delta$ in the rotating frame, but its thermal occupation number is still n_{b0}. Hence, the LC oscillator can now be viewed as being in contact with a cold reservoir with the effective temperature

$$T_{eff} = T_0\frac{|\Delta|}{\omega_b} \ll T_0, \tag{12}$$

so that it can extract energy from the nanomechanical mode via the coupling. In other words, the periodical modulation of the coupling up-converts the low-frequency nanomechanical quanta to the high-frequency LC oscillator quanta. This can be compared to the sympathetic-cooling scheme where one ion (atom) species at a lower temperature is used to cool another ion (atom) species at a higher temperature via Coulomb coupling.[35]

The above cooling scheme can also be viewed as an analog of the laser-cooling scheme in the resolved-sideband regime.[28] With periodical modulation of the linear coupling at the detuning $-\Delta = \omega_a$, the term $a^\dagger b + b^\dagger a$ generates a resonant cooling transition with the cooling rate $A_- = 4g_l^2/(\hbar^2\kappa_0)$. At the same time, the counterrotating term $a^\dagger b^\dagger + ba$ generates an off-resonant heating transition with the heating rate $A_+ \approx g_l^2\kappa_0/(4\hbar^2\omega_a^2)$. This analysis also shows that the counterrotating term is the origin of the quantum back-action noise[23] and the cooling is limited by the occupation number $n_0 = A_+/(A_- - A_+) \approx \kappa_0^2/16\omega_a^2$, agreeing with our previous result.

In fact, if we *omit* the counterrotating term when studying the cooling process, we can derive the exact final occupation number using a master-equation approach

$$n_a^f = (1 + [4(\omega_a + \Delta)^2 + \kappa_0^2]\hbar^2/4g_l^2)(\gamma_m/\kappa_0)n_{a0} \tag{13}$$

to the first order of γ_m, as is also plotted in Fig. 3. In contrast to the result from the quantum theory, the occupation number here is solely limited by the damping rate γ_m without the n_0 term. As the mechanical quality factor Q_m increases, n_a^f decreases accordingly.

V. REALIZATION

One experimental system to implement this scheme is a nanomechanical resonator capacitively coupling with a superconducting LC resonator.[31] Both the nanomechanical resonator and LC resonator have been tested in a wide range of parameters.[5,13,14] Here, a set of realistic parameters for the capacitances are $C_x' = 1$ fF/μm, $C_{x0} = 0.2$ fF, and $C_{\Sigma 0} = 1.5$ fF. For the LC oscillator, we have $\omega_b = 2\pi \times 7.5$ GHz and the damping rate $\kappa_0 = 2\pi \times 1$ MHz (or $Q = 7500$). For the nanomechanical mode, we choose $\omega_a = 2\pi \times 5$ MHz in our calculation and mechanical quality factors exceeding $Q_m = 3 \times 10^5$ have been demonstrated. At $T_0 = 20$ mK, the thermal occupation number is $n_{a0} = 83.2$. Given a gate voltage $v_c = 300$ mV, which corresponds to a reactive driving power of $P = v_c^2 C_x \omega_b \sim 0.850$ μW, the coupling constant is $g_l/\hbar = 2\pi \times 0.55$ MHz. As is shown in Fig. 3, the occupation number of the nanomechanical mode can be cooled to $n_a^f = 0.036$ at $Q_m = 10^4$. As Q_m increases, the occupation number will reach $n_0 = 0.0025$, only limited by the effect of the quantum back-action noise. This shows that ground-state cooling of the nanomechanical mode can be achieved in this system.

In previous schemes using superconducting circuits, the nanomechanical resonator couples with a LC oscillator via radiation-pressure-like coupling. The circuit is shown schematically in Fig. 1(b), which has been studied experimentally,[4,16] and the resolved-sideband regime has been reached at $\omega_a \sim 2\pi \times 1.5$ MHz.[5] In circuit (b), the coupling has the form of $-g_r(a+a^\dagger)b^\dagger b$, with the coupling constant $g_r=(\hbar\omega_b/2)(C_x' x_0/C_{\Sigma 0})$. The cooling rate of circuit (b) is given by Eq. (10) with g_l^2 replaced by $g_r^2\bar{n}_b$, where $\bar{n}_b$ is the average occupation number of the LC oscillator by the microwave driving v_g. For the cooling rate to be comparable to that of circuit (a), where $\bar{n}_b \approx 0$, it requires that $\bar{n}_b \approx 5.5 \times 10^7$ in circuit (b) which can result in strong nonlinear effect or additional damping in the circuit.

VI. CONCLUSIONS

To conclude, we studied a mechanism for the cooling of a nanomechanical resonator approaching the ground state. The scheme can be realized in a superconducting circuit with realistic parameters and can avoid the requirement of large occupation of the LC oscillator in previous schemes. Meanwhile, we also illustrated that the origin of the quantum back-action noise is the counterrotating term in the coupling.

The author is grateful to Mark Kasevich and Dan M. Stamper-Kurn for stimulating discussions and to GRC funds of UC Merced for support.

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